Exit-time of an inhomogeneous diffusion

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Abstract

We investigate the exit problem for a diffusion which drift is not time-homogeneous. More precisely, we study this problem for a McKean-Vlasov diffusion, that corresponds to the probabilistic interpretation of the granular media equation. This problem has already been solved in previous articles when the confining potential is uniformly strictly convex. Two different methods have been used. However, these two methods do not extend to the non-convex case. Consequently, here, we proceed in another way: by making a coupling with another McKean-Vlasov diffusion with a uniformly strictly convex confining potential. We present the result in a simple case, the one in which the interacting potential is linear. However, the result can be extended in a more general setting.

Key words and phrases: Self-Stabilizing diffusions ; Exit-time ; Granular media equation ; Large deviations ; Freidlin and Wentzell theory ; Interacting particle systems ; Coupling method

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1 Introduction

We study the exit problem of the diffusion X^σ defined by

$$X_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla V\left(X_s^\sigma\right) ds - \alpha \int_0^t \left(X_s^\sigma - \mathbb{E}\left[X_s^\sigma\right]\right) ds\,. \tag{1}$$

Here, σ is a positive constant which is arbitrarily small, x_0 is a real, B is a Brownian motion and α is a real - non necessarily positive. The exact hypotheses

*Université Jean Monnet, Saint-Étienne and Institut Camille Jordan, Lyon on V and F will be given subsequently in the introduction. For instance, we assume that V and F are $\mathcal{C}^2$-continuous.

Equation (1) corresponds to a McKean-Vlasov diffusion in which the confining potential is V and the interacting potential is $F(x) := \frac{\alpha}{2}x^2$. The diffusion can be written like so

$$\begin{cases} X_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla V\left(X_s^\sigma\right)ds - \int_0^t \nabla F * \mu_s^\sigma\left(X_s^\sigma\right)ds \\ \mu_t^\sigma = \mathcal{L}\left(X_t^\sigma\right) \end{cases}. \tag{2}$$

Here, $*$ denotes the convolution. Since the own law of the process intervenes in the drift, this equation is nonlinear - in the sense of McKean.

The motion of the process is generated by three concurrent forces. The first one is the derivative of the confining potential. The second influence is a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$. It allows the particle to move upwards the potential V. The third term - the so-called self-stabilizing term - represents the attraction with all the others trajectories. Indeed, we remark:

$$F' * \mu_s^\sigma\left(X_s(\omega_0)\right) = \int_{\omega\in\Omega} F'\left(X_s(\omega_0) - X_s(\omega)\right)d\mathbb{P}\left(\omega\right)$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying measurable space.

This diffusion corresponds to the probabilistic interpretation of a nonlinear partial differential equation, the granular media equation:

$$\frac{\partial u}{\partial t} = \nabla.\left\{\frac{\sigma^2}{2}\nabla u + \left(\nabla V + \nabla F * u\right)u\right\}. \tag{3}$$

Indeed, due to [McK66, McK67], the measure of probability $\mathcal{L}\left(X_t^\sigma\right)$ is absolutely continuous with respect to the Lebesgue measure whenever t is positive. By $u^\sigma\left(t,x\right)$, we denote its density. Then, $\{u^\sigma(t,x); t>0, x\in\mathbb{R}\}$ satisfies Equation (3).

The existence and the uniqueness of a strong solution on $\mathbb{R}_+$ for Equation (2) has been proved in [HIP08] (Theorem 2.13). The asymptotic behaviour of the law has been studied in [CGM08, BRV98] (for the convex case) and in [Tug13a, Tug13b] for the non-convex case by using the results in [HT10a, HT10b, HT12] about the non-uniqueness of the invariant probabilities and their small-noise behaviour.

The nonlinear diffusion (2) is obtained as the hydrodynamical limit of a mean-field system of interacting particles:

$$X_t^{i,N,\sigma} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N,\sigma}\right)ds - \int_0^t \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_s^{i,N,\sigma} - X_s^{j,N,\sigma}\right)ds\,. \tag{4}$$

Indeed, the previous equation can be written like so:

$$X_t^{i,N,\sigma} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N,\sigma}\right)ds - \int_0^t \nabla F * \eta_s^{N,\sigma}\left(X_s^{i,N,\sigma}\right)ds\,,$$

$\eta_s^{N,\sigma}$ being the empirical measure of the system: $\eta_s^{N,\sigma} := \frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,N,\sigma}}$. The particles are exchangeables and propagation of chaos implies that the diffusions $\left(\left(X_t^{i,N,\sigma}\right)_{t\geq 0}\right)_{1\leq i\leq N}$ become independent as N goes to infinity. So, intuitively, the empirical measure $\eta_t^{N,\sigma}$ converges to a measure $\eta_t^{\infty,\sigma}$ so that the drift in Equation (4) converges toward $\nabla V + \nabla F * \eta_t^{\infty,\sigma}$. Moreover, we have $\eta_t^{\infty,\sigma} = \mathcal{L}\left(X_t^{1,\infty,\sigma}\right) = \cdots = \mathcal{L}\left(X_t^{i,\infty,\sigma}\right)$ (for any $i \in \mathbb{N}^*$). We deduce that Equation (4) converges to

$$X_t^{i,\infty,\sigma} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,\infty,\sigma}\right) ds - \int_0^t \nabla F * \mathcal{L}\left(X_s^{i,\infty,\sigma}\right)\left(X_s^{i,\infty,\sigma}\right) ds\,.$$

This is exactly Equation (2).

In the setting of this paper, the mean-field system of interacting particles is

$$X_t^{i,\sigma} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,\sigma}\right) ds - \alpha \int_0^t \left(X_s^{i,\sigma} - \frac{1}{N}\sum_{j=1}^N X_s^{j,\sigma}\right) ds\,. \quad (5)$$

Let us present what we denote by exit problem. We consider a domain $\mathcal{D} \subset \mathbb{R}^d$ and we introduce

$$S_{X,\mathcal{D}} := \inf\left\{t \geq 0 \;\mid\; X_t^\sigma \in \mathcal{D}\right\}$$

the first hitting time of X^σ to the domain $\mathcal{D}$. Then, we define

$$\mathcal{T}_{X,\mathcal{D}} := \inf\left\{t \geq S_{X,\mathcal{D}} \;\mid\; X_t^\sigma \notin \mathcal{D}\right\}$$

the first exit-time of X^σ from the domain $\mathcal{D}$. The exit problem consists of two questions. What is the exit-time? What is the exit-location?
In the small-noise limit, the questions become:

1. What is the exit-time $\mathcal{T}_{X,\mathcal{D}}$ for σ going to 0?
2. What is the exit-location $X^\sigma_{\mathcal{T}_{X,\mathcal{D}}}$ for σ going to 0?

The subject of this article is to study these questions. In fact, we will only investigate the first one. Indeed, we can solve the exit-location question like in [Tug12], by using the results on the exit-time.

The natural framework is the one of the large deviations. Freidlin and Wentzell theory solves the exit problem for the time-homogeneous diffusions. Let us briefly present this theory. We refer the reader to [FW98, DZ98] for a complete review. We look at the diffusion

$$x_t^\sigma = x_0 + \sigma\beta_t - \int_0^t \nabla U\left(x_s^\sigma\right) ds\,.$$

U is a $\mathcal{C}^\infty$-continuous function from $\mathbb{R}^k$ ($k \geq 1$) to $\mathbb{R}$ and β is a Brownian motion in $\mathbb{R}^k$. Let a_0 be a minimizer of U and $\mathcal{G}$ be a domain which contains a_0.

We also consider the deterministic path $\Psi(x_0)$:

$$\Psi_t(x_0) = x_0 - \int_0^t \nabla U\left(\Psi_s(x_0)\right) \,.$$

Then:

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\sup_{t\in[0;T]}||x_t^\sigma - \Psi_t(x_0)|| > \delta\right\} = 0\,,$$

for any $T, \delta > 0$.

Moreover, under easily checked assumptions, for any $\delta > 0$, the following Kramers' type law holds:

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H-\delta\right)\right] < \mathcal{T}_{x,\mathcal{G}} < \exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right\} = 1\,.$$

Here, the exit cost is $H := \inf_{z\in\partial\mathcal{G}}\left[U(z) - U(a_0)\right]$. We immediately remark that

$$H = \lim_{\sigma\to 0}\frac{\sigma^2}{2}\log\left\{\mathbb{E}\left[\mathcal{T}_{x,\mathcal{G}}(\sigma)\right]\right\}\,.$$

Moreover, the exit-location is near the points of the boundary minimizing U. Indeed,

$$\lim_{\sigma\to 0}\mathbb{P}\left\{x^\sigma_{\mathcal{T}_{x,\mathcal{G}}}\in\mathcal{N}\right\} = 0$$

if $\mathcal{N}\subset\partial\mathcal{G}$ is such that $\inf_{z\in\mathcal{N}} U(z) > H$.

Let us now present the previous works on the exit problem of McKean-Vlasov diffusions.

Herrmann, Imkeller and Peithmann solve the problem when V is uniformly strictly convex. They adapt the Freidlin and Wentzell theory to the inhomogeneous diffusion. However, to establish their result, they need the diffusion $Y^{\sigma,\infty}$ to be an exponentially good approximation of the diffusion X^σ where $Y^{\sigma,\infty}$ is defined by

$$Y_t^{\sigma,\infty} = x_0 + \sigma B_t - \int_0^t \nabla V\left(Y_s^{\sigma,\infty}\right)ds - \int_0^t \nabla F\left(Y_s^{\sigma,\infty} - a\right)ds\,,$$

a being the unique wells of the potentiql V in the domain $\mathcal{G}$. To do so, they need the uniform convergence of $b^\sigma(t,x) := \nabla V(x) + \nabla F * \mu_t^\sigma(x)$ toward $b(t,x) := \nabla V(x) + \nabla F\left(x - \Psi_t(x_0)\right)$. And, this result requires the strict convexity of the potential V albeit the authors present it only under strict uniform convexity.

The problem in the previous method is linked to the long-time behaviour (the behaviour of $b^\sigma(t,x)$ when t is large). Thus, an idea is to study the long-time convergence of the drift $b^\sigma(t,x)$ toward a limit $b^\sigma(x)$ and the rate of this convergence. Indeed, if this rate of convergence does not depend on σ, one can solve the problem if moreover b^σ converges to b with $b(x) := \lim_{t\to\infty} b(t,x)$.

The convergence has been obtained in the convex case (see [BRV98, CMV03, CGM08, BCCP98]) and also in the non-convex case (see [Tug13a, Tug13b] albeit without the rate of convergence). Bolley, Gentil and Guillin has established the rate of convergence in [BGG12] in the convex case by using WJ-inequality. And, in the non-convex case, the rate of convergence has been obtained in [DMT13] by the same method. However, this result occurs only for the large values of σ. In fact, when the confining potential V has multiple wells, we may have, under easy to check assumptions:

$$b^\sigma(t,x) \longrightarrow b_1(t,x) \longrightarrow b_1(x)$$

and

$$b^\sigma(t,x) \longrightarrow b_2^\sigma(x) \longrightarrow b_2(x)$$

with $b_1 \neq b_2$. We deduce that under these assumptions, we can not get the uniform convergence of $b^\sigma(t,x)$ to $b(t,x)$.

This method thus seems inapplicable when the potentiel V is not convex. Let us briefly present the method in [Tug12]. We first solve the exit problem for the mean-field system of particles (5). More precisely, we establish result about the first exit-time of the first particle of this system. Then, we use a propagation of chaos of the form

$$\lim_{\substack{N\to\infty \\ \sigma\to 0}} \mathbb{E}\left\{\sup_{0\leq t\leq \mathcal{T}^N(\sigma)} \left|\left|X_t^\sigma - X_t^{1,\sigma,N}\right|\right|^2\right\} = 0\,,$$

where $\mathcal{T}^N(\sigma)$ is a stopping time which depends on the system of particles and on the McKean-Vlasov diffusion. However, to get this limit, we need μ_t^σ - the law of the inhomogeneous diffusion at time t - to be confined in a small ball around δ_a. This requires the convexity of the potential V. Consequently, we need to use another method.

In this paper, we simply make a coupling between the diffusion X^σ and another McKean-Vlasov diffusion,

$$Y_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla V_0\left(Y_s^\sigma\right) ds - \alpha \int_0^t \left(Y_s^\sigma - \mathbb{E}\left[Y_s^\sigma\right]\right) ds$$

where the confining potential V_0 is uniformly strictly convex and is equal to the potential V except on a compact domain $\mathcal{K}$. In the following, the domain from which we study the exit-time is $\mathcal{D} \subset \mathcal{K}^c$.

If the two diffusions were time-homogeneous, we could write

$$\nabla V\left(X_s^\sigma\right) \mathbb{1}_{s\leq \mathcal{T}_{X,\mathcal{D}}} = \nabla V_0\left(Y_s^\sigma\right) \mathbb{1}_{s\leq \mathcal{T}_{X,\mathcal{D}}}$$

so that $\mathcal{T}_{X,\mathcal{D}} = \mathcal{T}_{Y,\mathcal{D}}$. However, the two diffusions are inhomogeneous and it is a bit more difficult.

Before ending the introduction, we present the hypotheses. We say that the confining potential V satisfies the set of assumptions (A) if it satisfies the five next hypotheses:

(V-1) The potential V is $\mathcal{C}^2$-continuous.

(V-2) For any $\lambda > 0$, there exists R_λ such that $\nabla^2 V(x) \geq \lambda$ whenever $x \in \mathbb{R}$ satisfies $||x|| \geq R_\lambda$.

Immediately, we deduce that there exists a potential $\widetilde{V}$ which is strictly convex and such that $V = \widetilde{V}$ except on a compact domain. We assume also that this potential is uniformly strictly convex.

(V-3) There exists a uniformly strictly convex potential V_0 ($\nabla^2 V_0 \geq \theta > 0$) and a compact domain $\mathcal{K}$ such that $V(x) = V_0(x)$ for any $x \notin \mathcal{K}$.

(V-4) There exists $m \in \mathbb{N}^*$ and $C > 0$ such that $||\nabla V(x)|| \leq C\left(1 + ||x||^{2m-1}\right)$ for any $x \in \mathbb{R}$.

This assumption implies the existence of a solution to Equation (1) according to Proposition 2.13 in [HIP08].

(V-5) $\alpha + \theta > 0$, where θ is introduced in the hypothesis (V-3).

By $\mathcal{D}$, we denote the domain about which we search the exit-time. We assume $\mathcal{D} \subset \mathcal{K}^c$ and a_0 is the unique wells of V_0 on $\mathcal{D}$ (so the unique one of V on $\mathcal{D}$). In the following, we put

$$\psi := V - V_0\,.$$

Since ψ is $\mathcal{C}^2$-continuous and equal to zero except on the compact $\mathcal{K}$, its derivative is bounded:

$$\sup_{x\in\mathbb{R}} |\psi'(x)| \leq M\,.$$

The paper is organized as follows. In a first section, we present the basical results and definitions. Then, we provide an upperbound on the first hitting time of the domain $\mathcal{K}$. Finally, we give the main results.

2 Preliminaries

In this section, we present the definitions and the classical results that are used in this article.

Definition 2.1. *By Y^σ, we denote the McKean-Vlasov diffusion defined with the confining potential V_0 and the interacting potential $F(x) := \frac{\alpha}{2}x^2$ starting from x_0:*

$$Y_t^\sigma = x_0 + \sigma B_t - \int_0^t V_0'\left(Y_s^\sigma\right) ds - \alpha \int_0^t \left(Y_s^\sigma - \mathbb{E}\left[Y_s^\sigma\right]\right) ds\,.$$

We remind the reader that the potential V_0 is equal to the potential V except on the compact domain $\mathcal{K}$. Moreover, V_0 is uniformly strictly convex so we can apply the results in [HIP08] and in [Tug12].

Definition 2.2. *From now on, for any diffusion x^σ and for any domain $\mathcal{D}$, by $\mathcal{T}_{x,\mathcal{D}}$, we denote the first exit-time of the diffusion x^σ from the domain $\mathcal{D}$.*

Let us write here Theorem 4.1 in [Tug12] concerning the exit-time $\mathcal{T}_{Y,\mathcal{D}}$.

Proposition 2.3. *We assume that V satisfies the set of assumptions (A). Let us consider a domain $\mathcal{D}$ which satisfies the following assumptions:*

1. *The domain $\mathcal{D}$ is into $\mathcal{K}^c$.*
2. *For any $t \geq 0$, we have $\varphi_t(x_0) \in \mathcal{D}$ with $\varphi_t(x_0) = x_0 - \int_0^t V'(\varphi_s(x_0))\,ds$.*
3. *For any $t \geq 0$, for any $x \in \mathcal{D}$, we have $\psi_t(x) \in \mathcal{D}$ with $\psi_t(x) = x - \int_0^t \left[V'(\psi_s(x)) + F'(\psi_s(x) - a_0)\right] ds$.*

Then, for any $\xi > 0$, we have

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}(H-\xi)\right] \leq \mathcal{T}_{Y,\mathcal{D}} \leq \exp\left[\frac{2}{\sigma^2}(H+\xi)\right]\right\} = 1\,,$$

with $H := \inf_{z\in\partial\mathcal{D}} V(z) + \frac{\alpha}{2}(z-a_0)^2 = \inf_{z\in\partial\mathcal{D}} V_0(z) + \frac{\alpha}{2}(z-a_0)^2$.

We can apply this proposition to some particular domains, the level set domains.

Definition 2.4. *For any $H > 0$, we put*

$$\Lambda_H := \left\{z \in \mathbb{R} \;:\; V(z) + \frac{\alpha}{2}(z-a_0)^2 \leq V(a_0) + H\right\}\,.$$

This domain is not necessarily path-connected. By $\mathcal{L}_H$, we denote the path-connected subset of Λ_H which contains a_0.

We now apply Proposition 2.3 to the level set domains.

Corollary 2.5. *We assume that V satisfies the set of assumptions (A). Let H be any positive real such that $\mathcal{L}_H \bigcap \mathcal{K} = \emptyset$. Then, for any $\xi > 0$, we have*

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}(H-\xi)\right] \leq \mathcal{T}_{Y,\mathcal{L}_H} \leq \exp\left[\frac{2}{\sigma^2}(H+\xi)\right]\right\} = 1\,.$$

We give a last definition concerning the level set domains.

Definition 2.6. *We put*

$$d_H := \inf_{z\in\mathcal{L}_H}\inf_{x\in\mathcal{K}} |x-z|\,.$$

Intuitively, the diffusions X^σ and Y^σ are close if X has not reached the compact $\mathcal{K}$. However, we can not write $X_t^\sigma \mathbb{1}_{t\leq\mathcal{T}_{X,\mathcal{K}^c}} = Y_t^\sigma \mathbb{1}_{t\leq\mathcal{T}_{X,\mathcal{K}^c}}$. Indeed, the expectation of X_t^σ intervenes in the drift and we have $\mathbb{P}(t \leq \mathcal{T}_{X,\mathcal{K}^c}) < 1$ whenever t is positive.
Consequently, the hitting time of the diffusion X^σ to the domain $\mathcal{K}$ plays a big role in this paper.

Definition 2.7. *By $\mathcal{T}_0 := \mathcal{T}_{X,\mathcal{K}^c}$, we denote the first time that the McKean-Vlasov diffusion X^σ hits the compact $\mathcal{K}$.*

According to Proposition 2.3, we can remark that:

$$\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H(\mathcal{K}^c)-\delta\right)\right]\leq\mathcal{T}_{Y,\mathcal{K}^c}\leq\exp\left[\frac{2}{\sigma^2}\left(H(\mathcal{K}^c)+\delta\right)\right]\right\}\longrightarrow 1$$

as σ goes toward 0. This limit holds for any $\delta>0$. Here, $H(\mathcal{K}^c)$ is the infimum of the potential $z\mapsto V_0(z)+\frac{\alpha}{2}(z-a_0)^2=V(z)+\frac{\alpha}{2}(z-a_0)^2$. The infimum runs over the boundary of the path-connected subset of $\mathcal{K}^c$ which contains a_0.
We expect a similar Kramer's type law for $\mathcal{T}_0$.

3 Upper-bound for $\mathcal{T}_0$ and coupling result

As we have seen in the previous section, the hitting time of the domain $\mathcal{K}$ for the diffusion X^σ is important. The aim of this section is to provide an upper-bound to this hitting time $\mathcal{T}_0$ and to obtain a coupling result between X^σ and Y^σ. We remind the reader that the function ψ is defined by $\psi:=V-V_0$ and its derivative is bounded: $M:=\sup_{x\in\mathcal{K}}|\psi'(x)|<\infty$.
The first result establishes a link between the time $\mathcal{T}_0$ and the coupling.

Lemma 3.1. *Let us assume that V satisfies the set of assumptions (A). Then, for any positive t, we have:*

$$\mathbb{E}\left[||X_t^\sigma-Y_t^\sigma||^2\right]\leq\frac{M^2}{\theta^2}\mathbb{P}\left(\mathcal{T}_0\leq t\right)\,. \tag{6}$$

Proof. By using differential calculus, we have:

$$\begin{aligned}d\left(X_t^\sigma-Y_t^\sigma\right)^2=&-2\left\langle V'\left(X_t^\sigma\right)-V_0'\left(Y_t^\sigma\right)\,;\,X_t^\sigma-Y_t^\sigma\right\rangle dt\\&-2\alpha\left\langle X_t^\sigma-Y_t^\sigma\,;\,X_t^\sigma-Y_t^\sigma\right\rangle dt\\&+2\alpha\left\langle\mathbb{E}\left[X_t^\sigma-Y_t^\sigma\right]\,;\,X_t^\sigma-Y_t^\sigma\right\rangle dt\,.\end{aligned}$$

We now use the function ψ:

$$\begin{aligned}d\left(X_t^\sigma-Y_t^\sigma\right)^2=&-2\left\langle V_0'\left(X_t^\sigma\right)-V_0'\left(Y_t^\sigma\right)\,;\,X_t^\sigma-Y_t^\sigma\right\rangle dt\\&-2\alpha\left\langle X_t^\sigma-Y_t^\sigma\,;\,X_t^\sigma-Y_t^\sigma\right\rangle dt\\&+2\alpha\left\langle\mathbb{E}\left[X_t^\sigma-Y_t^\sigma\right]\,;\,X_t^\sigma-Y_t^\sigma\right\rangle dt\\&-2\left\langle\psi'\left(X_t^\sigma\right)\,;\,X_t^\sigma-Y_t^\sigma\right\rangle dt\,.\end{aligned}$$

By integrating, we obtain:

$$\begin{aligned}(X_t^\sigma - Y_t^\sigma)^2 = & -2\int_0^t \langle V_0'(X_s^\sigma) - V_0'(Y_s^\sigma)\ ;\ X_s^\sigma - Y_s^\sigma\rangle\, ds \\ & - 2\alpha\int_0^t \langle X_s^\sigma - Y_s^\sigma\ ;\ X_s^\sigma - Y_s^\sigma\rangle\, ds \\ & + 2\alpha\int_0^t \langle \mathbb{E}\left[X_s^\sigma - Y_s^\sigma\right]\ ;\ X_s^\sigma - Y_s^\sigma\rangle\, ds \\ & - 2\int_0^t \langle \psi'(X_s^\sigma)\ ;\ X_s^\sigma - Y_s^\sigma\rangle\, ds\,.\end{aligned}$$

We now take the expectation. The inequality

$$\mathbb{E}\left[(X_s^\sigma - Y_s^\sigma)^2\right] - \left(\mathbb{E}\left[X_s^\sigma - Y_s^\sigma\right]\right)^2 \geq 0$$

leads us to:

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right] \leq & -2\mathbb{E}\left[\langle V_0'(X_t^\sigma) - V_0'(Y_t^\sigma)\ ;\ X_t^\sigma - Y_t^\sigma\rangle\right] \\ & - 2\mathbb{E}\left[\langle \psi'(X_t^\sigma)\ ;\ X_t^\sigma - Y_t^\sigma\rangle\right]\,.\end{aligned}$$

By definition of the potential V_0, we can write $\psi' = \psi'\mathbb{1}_{\mathcal{K}}$ so that Cauchy-Schwarz inequality implies

$$\begin{aligned}-2\mathbb{E}\left[\langle \psi'(X_t^\sigma)\ ;\ X_t^\sigma - Y_t^\sigma\rangle\right] & \leq 2M\sqrt{\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right]}\sqrt{\mathbb{P}\left(X_t^\sigma \in \mathcal{K}\right)} \\ & \leq 2M\sqrt{\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right]}\sqrt{\mathbb{P}\left(\mathcal{T}_0 \leq t\right)}\,.\end{aligned}$$

Since $V_0'' \geq \theta > 0$, we deduce

$$\frac{d}{dt}\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right] \leq -2\theta\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right] + 2M\sqrt{\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right]}\sqrt{\mathbb{P}\left(\mathcal{T}_0 \leq t\right)}\,.$$

By putting $f(t) := \mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right]$, this means that

$$\{t \geq 0\ :\ f'(t) \leq 0\} \supset \left\{t \geq 0\ :\ f(t) \geq \frac{M^2}{\theta^2}\mathbb{P}\left(\mathcal{T}_0 \leq t\right)\right\}\,.$$

Recalling that $f(0) = 0$, this allows us to conclude that f is bounded by $\frac{M^2}{\theta^2}\mathbb{P}\left(\mathcal{T}_0 \leq t\right)$. $\square$

Consequently, if $\mathbb{P}\left(\mathcal{T}_0 \leq t\right)$ is small, we have a coupling between the two diffusions X^σ and Y^σ. However, we want to use the exit times of Y^σ to obtain results on the ones of X^σ. Consequently, we need a coupling which is uniform in time, the supremum being under the expectation. This is the aim of the following lemma.

Lemma 3.2. *Let us assume that V satisfies the set of assumptions (A). Then, for any positive t, we have:*

$$\mathbb{E}\left\{\sup_{t\leq\mathcal{T}_0}||X_t^\sigma - Y_t^\sigma||^2 \mathbb{1}_{\mathcal{T}_0\leq e^{\frac{2\Lambda}{\sigma^2}}}\right\} \leq \left(\frac{M\alpha}{\theta(\alpha+\theta)}\right)^2 \mathbb{P}\left(\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}\right), \qquad (7)$$

whenever Λ is positive.

Proof. Let t be a positive real such that $t < \mathcal{T}_0$. We deduce $V'(X_t^\sigma) = V'(X_t^\sigma)\mathbb{1}_{X_t^\sigma\in\mathcal{K}} = V_0'(X_t^\sigma)\mathbb{1}_{X_t^\sigma\in\mathcal{K}} = V_0'(X_t^\sigma)$. Differential calculus implies

$$\begin{aligned}\frac{d}{dt}(X_t^\sigma - Y_t^\sigma)^2 &= -2\left(V_0'(X_t^\sigma) - V_0'(Y_t^\sigma)\right)(X_t^\sigma - Y_t^\sigma)\\ &\quad -2\alpha(X_t^\sigma - Y_t^\sigma)^2 + 2\alpha\mathbb{E}[X_t^\sigma - Y_t^\sigma](X_t^\sigma - Y_t^\sigma)\\ &\leq -2(\alpha+\theta)(X_t^\sigma - Y_t^\sigma)^2 + 2\alpha\sqrt{\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right]}\,|X_t^\sigma - Y_t^\sigma|,\end{aligned}$$

after using Jensen inequality. By an argument similar to the one of the proof of Lemma 3.1, we deduce the following boundedness:

$$(X_t^\sigma - Y_t^\sigma)^2 \leq \left(\frac{\alpha}{\alpha+\theta}\right)^2 \mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right], \qquad (8)$$

for any $t \leq \mathcal{T}_0$. We now take the supremum over $[0;\mathcal{T}_0]$:

$$\sup_{t\leq\mathcal{T}_0}(X_t^\sigma - Y_t^\sigma)^2 \leq \left(\frac{\alpha}{\alpha+\theta}\right)^2 \sup_{t\leq\mathcal{T}_0}\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right].$$

Multiplying by $\mathbb{1}_{\mathcal{T}_0\leq e^{\frac{2\Lambda}{\sigma^2}}}$ yields

$$\mathbb{1}_{\mathcal{T}_0\leq e^{\frac{2\Lambda}{\sigma^2}}}\sup_{t\leq\mathcal{T}_0}(X_t^\sigma - Y_t^\sigma)^2 \leq \mathbb{1}_{\mathcal{T}_0\leq e^{\frac{2\Lambda}{\sigma^2}}}\left(\frac{\alpha}{\alpha+\theta}\right)^2 \sup_{t\leq e^{\frac{2\Lambda}{\sigma^2}}}\mathbb{E}\left[(X_t^\sigma - Y_t^\sigma)^2\right].$$

Taking the expectation and applying Inequality (6) ends the proof of Inequality (7). □

Proposition 3.3. *Let us assume that V satisfies the set of assumptions (A). Let Λ be a positive real such that $\mathcal{L}_\Lambda \subset \mathcal{K}^c$ and such that the following inequality holds:*

$$\frac{M|\alpha|}{\theta|\alpha+\theta|d_\Lambda} < 1.$$

Then, we have the following upperbound for the hitting time $\mathcal{T}_0$:

$$\lim_{\sigma\to0}\mathbb{P}\left(\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}\right) = 0. \qquad (9)$$

Proof. For any positive real δ such that $\mathcal{L}_{\Lambda+\delta} \subset \mathcal{K}^c$, one has

$$\begin{aligned}\mathbb{P}\left(\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}\right) =& \mathbb{P}\left\{\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}} \,;\, \mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}} \leq e^{\frac{2\Lambda}{\sigma^2}}\right\} + \mathbb{P}\left\{\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}} \,;\, \mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}} \geq e^{\frac{2\Lambda}{\sigma^2}}\right\}\\ \leq& \mathbb{P}\left\{\mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}} \leq e^{\frac{2\Lambda}{\sigma^2}}\right\} + \mathbb{P}\left\{\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}} \,;\, \mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}} \geq e^{\frac{2\Lambda}{\sigma^2}}\right\}.\end{aligned}$$

According to Corollary 2.5, the term $\mathbb{P}\left\{\mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}} \leq e^{\frac{2\Lambda}{\sigma^2}}\right\}$ goes to 0 as σ goes to 0. We now deal with the second term. We take δ sufficiently small such that $d_{\Lambda+\delta} > 0$.

$$\mathbb{P}\left\{\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}} \,;\, \mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}} \geq e^{\frac{2\Lambda}{\sigma^2}}\right\} \leq \mathbb{P}\left\{\sup_{t\leq\mathcal{T}_0} |X_t^\sigma - Y_t^\sigma| \geq d_{\Lambda+\delta} \,;\, \mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}\right\}.$$

Indeed, if we have $\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}} \leq \mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}}$, then $X^\sigma_{\mathcal{T}_0} \in \mathcal{K}$ and $Y^\sigma_{\mathcal{T}_0} \in \mathcal{L}_{\Lambda+\delta}$.
By using Inequality (7), we obtain

$$\mathbb{P}\left\{\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}} \,;\, \mathcal{T}_{Y,\mathcal{L}_{\Lambda+\delta}} \geq e^{\frac{2\Lambda}{\sigma^2}}\right\} \leq \frac{1}{d^2_{\Lambda+\delta}}\left(\frac{M\alpha}{\theta(\alpha+\theta)}\right)^2 \mathbb{P}\left(\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}\right).$$

By taking δ sufficiently small, we have $\frac{M|\alpha|}{\theta(\alpha+\theta)d_{\Lambda+\delta}} < 1$. This allows us to conclude that $\mathbb{P}\left(\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}\right)$ converges toward 0 as σ goes to 0. $\square$

We immediately deduce the following corollary.

Corollary 3.4. *Let us assume that V satisfies the set of assumptions (A). Let Λ be a positive real such that $\mathcal{L}_\Lambda \subset \mathcal{K}^c$ and such that the following inequality holds:*

$$\frac{M|\alpha|}{\theta|\alpha+\theta|d_\Lambda} < 1\,.$$

Then, we have the following coupling results:

$$\mathbb{E}\left\{\sup_{t\leq\mathcal{T}_0} ||X_t^\sigma - Y_t^\sigma||^2 \, \mathbb{1}_{\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}}\right\} \longrightarrow 0$$

and

$$\sup_{t\leq e^{\frac{2\Lambda}{\sigma^2}}} \mathbb{E}\left[||X_t^\sigma - Y_t^\sigma||^2\right] \longrightarrow 0\,.$$

The proofs of the two inequalities are left to the reader. We also have the following useful corollary.

Corollary 3.5. *Let us assume that V satisfies the set of assumptions (A). Let Λ be a positive real such that $\mathcal{L}_\Lambda \subset \mathcal{K}^c$ and such that the following inequality holds:*

$$\frac{M|\alpha|}{\theta|\alpha+\theta|d_\Lambda} < 1\,.$$

Then, the following limit is true:

$$\mathbb{E}\left\{\sup_{t\leq\min\left\{\mathcal{T}_0;e^{\frac{2\Lambda}{\sigma^2}}\right\}}||X_t^\sigma-Y_t^\sigma||^2\right\}\longrightarrow 0 \tag{10}$$

as σ goes to 0.

Proof. We remind the reader Inequality (8):

$$(X_t^\sigma-Y_t^\sigma)^2\leq\left(\frac{\alpha}{\alpha+\theta}\right)^2\mathbb{E}\left[(X_t^\sigma-Y_t^\sigma)^2\right]\,,$$

for any $t\leq\mathcal{T}_0$. We take the supremum over $\left[0;\min\left\{\mathcal{T}_0;e^{\frac{2\Lambda}{\sigma^2}}\right\}\right]$ and we obtain

$$\begin{aligned}\sup_{t\leq\min\left\{\mathcal{T}_0;e^{\frac{2\Lambda}{\sigma^2}}\right\}}(X_t^\sigma-Y_t^\sigma)^2&\leq\left(\frac{\alpha}{\alpha+\theta}\right)^2\sup_{t\leq\min\left\{\mathcal{T}_0;e^{\frac{2\Lambda}{\sigma^2}}\right\}}\mathbb{E}\left[(X_t^\sigma-Y_t^\sigma)^2\right]\\&\leq\left(\frac{\alpha}{\alpha+\theta}\right)^2\sup_{t\leq e^{\frac{2\Lambda}{\sigma^2}}}\mathbb{E}\left[(X_t^\sigma-Y_t^\sigma)^2\right]\,.\end{aligned}$$

We take the expectation. Then, according to Corollary 3.4, this converges toward 0 as σ goes to 0. □

4 Main results

We now present the two main results of the paper concerning the exit time of the diffusion X^σ from a domain $\mathcal{D}$ satisfying the usual conditions plus one condition about its exit cost.

Definition 4.1. *By Λ_0, we denote the supremum of the positive Λ such that $\frac{M|\alpha|}{\theta(\alpha+\theta)d_\Lambda}<1$.*

Let us remark that such positive Λ satisfies $d_\Lambda>0$ so that $\mathcal{L}_\Lambda\subset\mathcal{K}^c$.

Immediately, we deduce that for any $\Lambda<\Lambda_0$, we have the limit:

$$\lim_{\sigma\to0}\mathbb{E}\left\{\sup_{t\leq\min\left\{\mathcal{T}_0;e^{\frac{2\Lambda}{\sigma^2}}\right\}}||X_t^\sigma-Y_t^\sigma||^2\right\}=0\,,$$

thanks to Corollary 3.5.

Theorem 4.2. *Let us assume that V satisfies the set of assumptions (A). Let $\mathcal{D}$ be an open domain satisfying the following assumptions:*

- *The domain $\mathcal{D}$ is into the domain $\mathcal{K}^c$.*
- *For any $t \geq 0$, we have $\varphi_t(x_0) \in \mathcal{D}$ with $\varphi_t(x_0) = x_0 - \int_0^t V'\left(\varphi_s(x_0)\right) ds$.*
- *For any $t \geq 0$, for any $x \in \mathcal{D}$, we have $\psi_t(x) \in \mathcal{D}$ with $\psi_t(x) = x - \int_0^t \left[V'\left(\psi_s(x)\right) + F'\left(\psi_s(x) - a_0\right)\right] ds$.*

We also assume that its exit cost

$$H := \inf_{z \in \partial\mathcal{D}} \left[V(z) - V(a_0)\right] = \inf_{z \in \partial\mathcal{D}} \left[V_0(z) - V_0(a_0)\right]$$

satisfies $H < \Lambda_0$.
Then, we have a Kramer's type law. In other words, for any $\delta > 0$, we have the following limit as σ goes to 0:

$$\mathbb{P}\left\{\exp\left[\frac{2(H-\delta)}{\sigma^2}\right] \leq \mathcal{T}_{X,\mathcal{D}} \leq \exp\left[\frac{2(H+\delta)}{\sigma^2}\right]\right\} \longrightarrow 1 \tag{11}$$

Proof. Let δ be any positive real.
We first prove the lower-bound. We have immediately

$$\begin{aligned}
\mathbb{P}\left(\mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\right) =& \mathbb{P}\left\{\mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\,;\, \mathcal{T}_{Y,\mathcal{L}_{H-\frac{\delta}{2}}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\right\} \\
&+ \mathbb{P}\left\{\mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\,;\, \mathcal{T}_{Y,\mathcal{L}_{H-\frac{\delta}{2}}} \geq e^{\frac{2}{\sigma^2}(H-\delta)}\right\} \\
\leq& \mathbb{P}\left\{\mathcal{T}_{Y,\mathcal{L}_{H-\frac{\delta}{2}}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\right\} \\
&+ \mathbb{P}\left\{\mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\,;\, \mathcal{T}_{Y,\mathcal{L}_{H-\frac{\delta}{2}}} \geq e^{\frac{2}{\sigma^2}(H-\delta)}\right\}\,.
\end{aligned}$$

According to Corollary 2.5, the first term goes to 0 as σ goes to 0.
We now deal with the second term. Due to the definition of H, we deduce that the domain $\mathcal{L}_{H-\frac{\delta}{2}}$ is into the domain $\mathcal{D}$. We put $\rho(\delta) := \mathrm{d}\left(\mathcal{L}_{H+\frac{\delta}{2}}\,;\, \mathcal{D}^c\right) > 0$. We deduce

$$\begin{aligned}
&\mathbb{P}\left\{\mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\,;\, \mathcal{T}_{Y,\mathcal{L}_{H-\frac{\delta}{2}}} \geq e^{\frac{2}{\sigma^2}(H-\delta)}\right\} \\
&\leq \mathbb{P}\left\{\sup_{t \leq \mathcal{T}_{X,\mathcal{D}}} |X_t^\sigma - Y_t^\sigma| \geq \rho(\delta)\,;\, \mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\right\}\,.
\end{aligned}$$

Indeed, if we have $\mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)} \leq \mathcal{T}_{Y,\mathcal{L}_{H-\frac{\delta}{2}}}$, then $X^\sigma_{\mathcal{T}_{X,\mathcal{D}}} \in \overline{\mathcal{D}}$ and $Y^\sigma_{\mathcal{T}_{X,\mathcal{D}}} \in \mathcal{L}_{H-\frac{\delta}{2}}$. However, the domain $\mathcal{D}$ is included into the domain $\mathcal{K}^c$. Consequently, we have:

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t \leq \mathcal{T}_{X,\mathcal{D}}} |X_t^\sigma - Y_t^\sigma| \geq \rho(\delta)\,;\, \mathcal{T}_{X,\mathcal{D}} \leq e^{\frac{2}{\sigma^2}(H-\delta)}\right\} \\
&\leq \frac{1}{\rho(\delta)^2}\mathbb{E}\left\{\sup_{t \leq e^{\frac{2}{\sigma^2}(H-\delta)}} |X_t^\sigma - Y_t^\sigma|^2\right\}
\end{aligned}$$

which converges toward 0 as σ goes to 0.
To prove the upper-bound, we can not consider the domain $\mathcal{L}_{H+\frac{\delta}{2}}$ because this domain does not contain $\mathcal{D}$ in its interior. Instead, let us consider an open domain $\mathcal{D}_\kappa$ which contains $\mathcal{D}$ in its interior and which satisfies the five following properties

- The domain $\mathcal{D}_\kappa$ is into the domain $\mathcal{K}^c$.
- For any $t \geq 0$, we have $\varphi_t(x_0) \in \mathcal{D}_\kappa$.
- For any $t \geq 0$, for any $x \in \mathcal{D}_\kappa$, we have $\psi_t(x) \in \mathcal{D}_\kappa$.
- The distance between $\mathcal{D}$ and $\mathcal{D}_\kappa^c$ is $\kappa > 0$.
- Its exit cost $\inf\limits_{z \in \mathcal{D}_\kappa} [V(z) - V(a_0)]$ is equal to $H + \frac{\delta}{2}$.

The existence of such a domain is ensured by Section 2 in [Tug12]. We thus proceed like for the proof of the lower-bound which achieves the proof of the theorem. $\square$

We deduce the following corollary.

Corollary 4.3. *Let us assume that V satisfies the set of assumptions (A). Let $\mathcal{D}$ be an open domain satisfying the following assumptions:*

- *The domain $\mathcal{D}$ is into the domain $\mathcal{K}^c$.*
- *For any $t \geq 0$, we have $\varphi_t(x_0) \in \mathcal{D}$ with $\varphi_t(x_0) = x_0 - \int_0^t V'(\varphi_s(x_0))\, ds$.*
- *For any $t \geq 0$, for any $x \in \mathcal{D}$, we have $\psi_t(x) \in \mathcal{D}$ with $\psi_t(x) = x - \int_0^t [V'(\psi_s(x)) + F'(\psi_s(x) - a_0)]\, ds$.*

Then, there exist $\alpha_- < 0 < \alpha_+$ such that whenever α is between α_- and α_+, we have a Kramer's type law. In other words, for any $\delta > 0$, we have Limit (11).

Remark 4.4. *We can solve the exit-location question by proceeding exactly like in [Tug12]. This is why we do not give it here. Typically, if $\mathcal{N}$ is a subset of $\mathcal{D}$ (where $\mathcal{D}$ satisfies the assumptions of Theorem 4.2) such that $\inf\limits_{z\in\mathcal{N}} V(z) > \inf\limits_{z\in\partial\mathcal{D}} V(z)$, the probability to exit $\mathcal{D}$ by the domain $\mathcal{N}$ goes to zero as σ goes to zero.*

Remark 4.5. *We have obtained a Kramer's type law for the hydrodynamical limit. And, the coupling result of Section 2 gives a uniform control of the moments of the diffusion X^σ. Consequently, by using the method developed in [Tug12], one can solve - under the same hypotheses - the exit-time question of the first particle in the mean-field system of particles.*

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